Entanglement witnesses: overview of the technique and a new construction

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Overview

Separable states and Schmidt decomposition

Definition: A pure state $\rho \in \mathcal{B}(\mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n})$ is called *separable* if it is represented by a product state vector:

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It generalises to decomposition of a bipartite state:

$$\rho = \sum_{i=1}^{\min(d_1^2,d_2^2)} \lambda_i F_i \otimes G_i, \qquad (3)$$

where $\langle F_i|F_j\rangle_{HS} = \delta_{ij}$, $\langle G_i|G_j\rangle_{HS} = \delta_{ij}$, $\sum_i \lambda_i^2 = 1$ and F_i's and G_i's are hermitian.

(not a separable decomposition - F_i and G_i in general not positive!)

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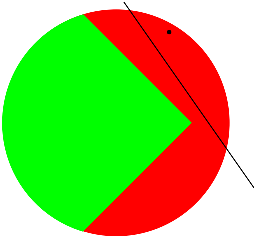

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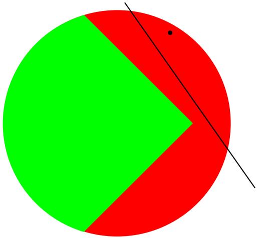

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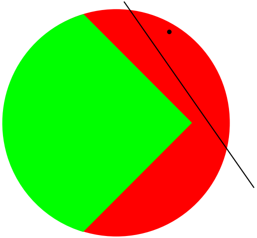

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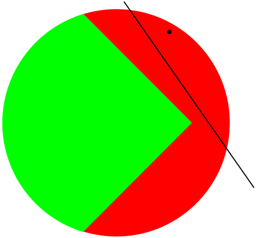

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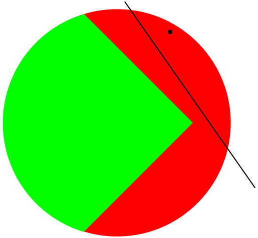

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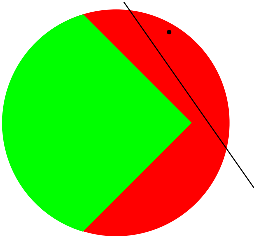

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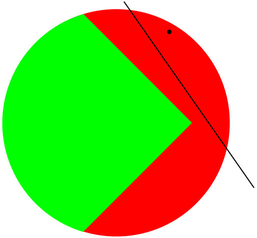

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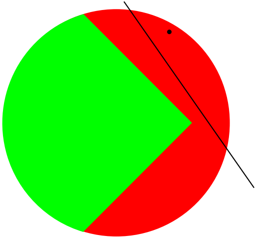

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Positivity of $(I_{d_1} \otimes \Phi)(\rho)$ is equivalent to positive expected value of a family of entanglement witnesses: $\{A \otimes BW_\Phi A^\dagger \otimes B^\dagger\}$.

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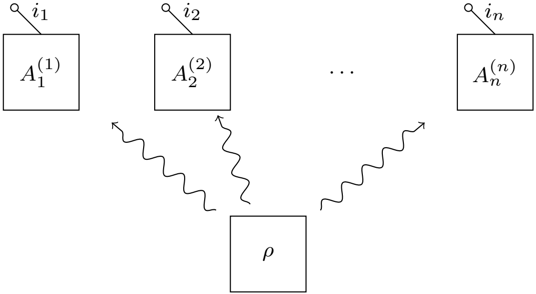

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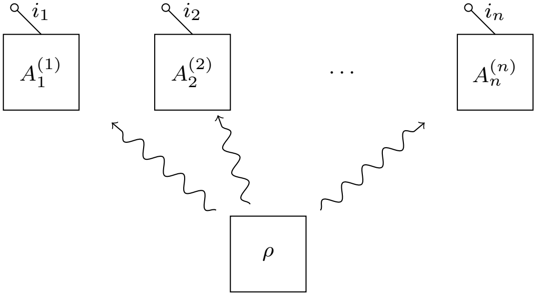

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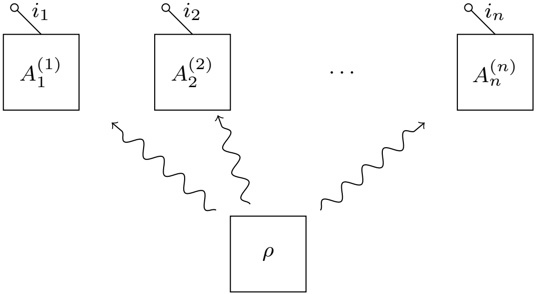

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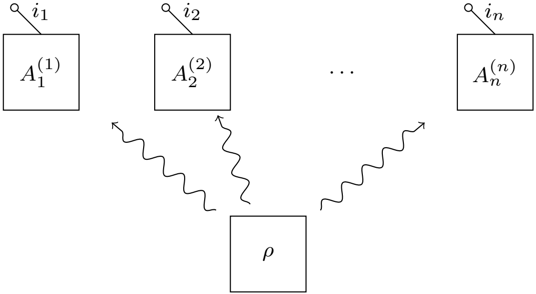

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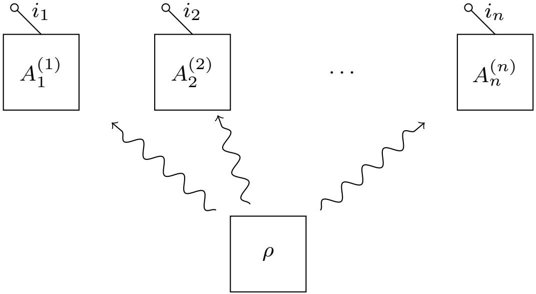

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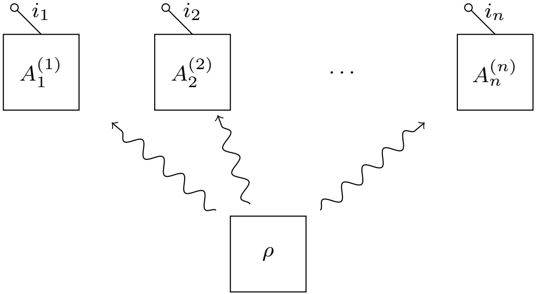

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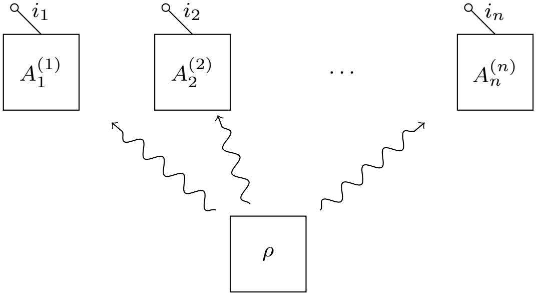

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Other criteria or other maps are necessary to detect such entanglement.

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- for positive A: $\|A\|_1 = \mathrm{TrA}$, in general: sum of singular values,
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Hence:

$$\|C(\rho)\|_1 = \max_{O \in O(d_1,d_2)} \mathrm{Tr}(\rho \sum_{ij} G_i^{(1)} \otimes G_j^{(2)} O^{ij}) \leq 1 \tag{6}$$

$$\forall O \in O(d_1,d_2)\ \mathrm{Tr}(\rho(I - \sum_{ij} G_i^{(1)} \otimes G_j^{(2)} O^{ij})) \geq 0 \tag{7}$$

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and the realignment criterion is equivalent to family of witnesses:

$$W_O = I - \sum_{ij} G_i^{(1)} \otimes G_j^{(2)} O^{ij}, \tag{8}$$

parametrised by isometry matrices.

Enhanced realignment criterion and other C-based criteria

These witnesses can be strengthen by a non-linear correction:

$$\widetilde{W}_O = I - \sum_{ij} G_i^{(1)} \otimes G_j^{(2)} O^{ij} - \frac{1}{2}(\mathrm{Tr}\rho_A^2 + \mathrm{Tr}\rho_B^2). \quad (9)$$

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- **ESIC criterion:** Let $\hat{C}_{ij} = \mathrm{Tr}(\rho P_i \otimes Q_j)$, where $\{P_i\}_{i=1}^{d_1^2}$ and $\{Q_i\}_{i=1}^{d_2^2}$ are SIC-POVMs. Then for ρ - separable:

$$\|\hat{C}\|_1 \leq \sqrt{\frac{d_1 + 1}{2d_1}}\sqrt{\frac{d_2 + 1}{2d_2}} \quad (12)$$

Generalisation of the linear criteria arXiv:2001.08258

Redefine: $C_{x,y} = \mathrm{diag}\{x, 1, \ldots, 1\} C \mathrm{diag}\{y, 1, \ldots, 1\}$.
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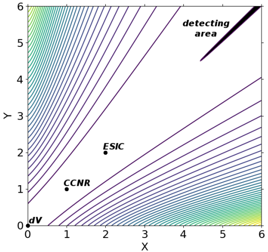

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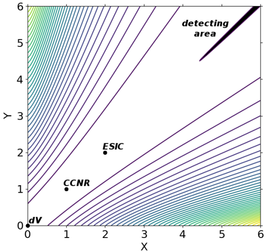

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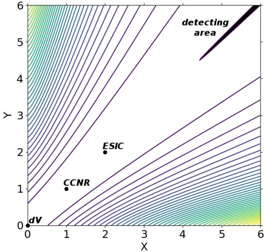

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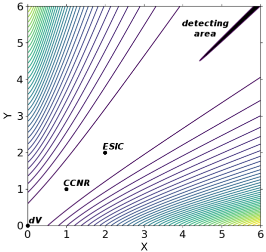

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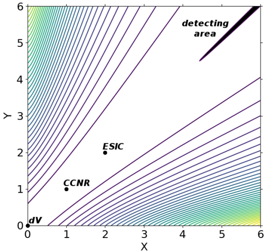

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We have proven the following: for ρ - separable:

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Limit of bipartite case arXiv:2002.00646

We find a family of witnesses corresponding to our criterion:

$$W_{O,x,y} = a_{xy} G_0^A \otimes G_0^B + x G_0^A \otimes \Big(\sum_{\beta>0} O^{0\beta} G_\beta^B \Big) + y \Big(\sum_{\beta>0} O^{\alpha 0} G_\alpha^A \Big) \otimes G_0^B + \sum_{\alpha,\beta>0} O^{\alpha\beta} G_\alpha^A \otimes G_\beta^B, \qquad (17)$$

where $a_{xy} = \left(\sqrt{d_A - 1 + x^2} \sqrt{d_B - 1 + y^2} + xyO^{00} \right)$. $\lim_{x,y\to\infty} O^{00} = -1$, otherwise $\lim W_{O,x,y} \sim I \otimes I$.
We take:

$$O = \left[\begin{array}{c|c} -\sqrt{1-\eta^2/r^2} & \eta/r \ \mathbf{v}^T \\ \hline \eta/r \ \mathbf{u} & \mathbf{O} \end{array} \right] \qquad (18)$$

(up to $O(r^2)$), where $\mathbf{u}$ and $\mathbf{v}$ are unit vectors satisfying $\mathbf{u} = \mathbf{O}\mathbf{v}/\sqrt{1-\eta^2/r^2} \xrightarrow{r\to\infty} \mathbf{O}\mathbf{v}$ and get the limit:

$$W^\infty = \frac{(d_B-1)\cot\theta + (d_A-1)\tan\theta + \eta^2 \sin\theta\cos\theta}{2} \frac{I_{d_A}}{\sqrt{d_A}} \otimes \frac{I_{d_B}}{\sqrt{d_B}} + \eta\cos\theta \frac{I_A}{\sqrt{d_A}} \otimes \sum_{\beta>0} v^\beta G_\beta^B$$

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Equivalence of criteria arXiv:2002.00646

First we prove, that (the simple part):

$$\|C(\rho - \rho_A \otimes \rho_B)\|_1 \leq \sqrt{1 - \mathrm{Tr}\rho_A^2}\sqrt{1 - \mathrm{Tr}\rho_B^2} \Rightarrow \|C_{xy}(\rho)\|_1 \leq \sqrt{\frac{d_A - 1 + x^2}{d_A}}\sqrt{\frac{d_B - 1 + y^2}{d_B}} \tag{20}$$

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and look for the minimum of their expected values for a given state ρ. It is attained for values of parameters:

- $\mathbf{O} = UV^T$, where UDV^T is a SVD of $\rho - \rho_A \otimes \rho_B$
- $\tan\theta = \sqrt{\frac{d_B(1-\|\rho_B\|^2)}{d_A(1-\|\rho_A\|^2)}}$
- $\eta = \frac{\sqrt{d_A d_B}}{\sin\theta\cos\theta}\left\|\frac{\cos\theta}{\sqrt{d_A}}\widetilde{\rho}_B + \frac{\sin\theta}{\sqrt{d_B}}\widetilde{O}^T\widetilde{\rho}_A\right\|$
- $v = -\frac{\frac{A\cos\theta}{\sqrt{d_A}}\widetilde{\rho}_B + \frac{A\sin\theta}{\sqrt{d_B}}\widetilde{O}^T\widetilde{\rho}_A}{\left\|\frac{A\cos\theta}{\sqrt{d_A}}\widetilde{\rho}_B + \frac{A\sin\theta}{\sqrt{d_B}}\widetilde{O}^T\widetilde{\rho}_A\right\|}$

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First we prove, that (the simple part):

$$\|C(\rho-\rho_A\otimes\rho_B)\|_1 \leq \sqrt{1-\mathrm{Tr}\rho_A^2}\sqrt{1-\mathrm{Tr}\rho_B^2} \Rightarrow \|C_{xy}(\rho)\|_1 \leq \sqrt{\frac{d_A-1+x^2}{d_A}}\sqrt{\frac{d_B-1+y^2}{d_B}} \tag{20}$$

for all x, y. Hence no correlation tensor based criterion can detect more that the enhanced realignment criterion. Now we consider the limit witnesses:

$$W^\infty = \frac{(d_B-1)\cot\theta+(d_A-1)\tan\theta+\eta^2\sin\theta\cos\theta}{2}\frac{I_{d_A}}{\sqrt{d_A}}\otimes\frac{I_{d_B}}{\sqrt{d_B}}+\eta\cos\theta\frac{I_A}{\sqrt{d_A}}\otimes\sum_{\beta>0}v^\beta G_\beta^B$$
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and look for the minimum of their expected values for a given state ρ. It is equal to:

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Hence if the enhanced realignment criterion detects entanglement in ρ, then it is detected by a witness of a form W^∞ as well, hence it is also detected by $W_{O,x,y}$ for large enough x, y.

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Thank you for your attention!

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